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14. ABSTRACT
This proposal concerns modeling, analysis, and computational simulations of some non-linear interface problems and applications, that, we believe, are of interests to DOD/ARO missions especially in the fields of computational geometry, computational fluid mechanics, non-linear dynamics, imaging processing, and inverse scattering.

Most of our previous work has been focused on linear interface problems

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Report Title

Final report of the project: Theoretical and Numerical Analysis for Non-linear Interface Problems

ABSTRACT

This proposal concerns modeling, analysis, and computational simulations of
some non-linear interface problems and applications, that, we believe,
are of interests to DOD/ARO missions especially in the fields of computational geometry, computational fluid mechanics, non-linear dynamics, imaging processing, and inverse scattering.

Most of our previous work has been focused on linear interface problems
with discontinuities. We have been focusing on non-linear interface problems and
applications that are of interests of DOD/ARO during the course of the project, particukarly for nonlinear interface problems in MR fluids and and weighted minimum surface problems.

We have successfully developed the immersed interface method for several non-linear interface problems including Poisson-Boltzmann equations on irregular domains, some variational problems, augmented approach for interface problems and problems on irregular domains.

List of papers submitted or published that acknowledge ARO support during this reporting period. List the papers, including journal references, in the following categories:

(a) Papers published in peer-reviewed journals (N/A for none)

Number of Papers published in peer-reviewed journals: 36.00

(b) Papers published in non-peer-reviewed journals or in conference proceedings (N/A for none)

Number of Papers published in non peer-reviewed journals: 4.00

(c) Presentations

Number of Presentations: 8.00

Non Peer-Reviewed Conference Proceeding publications (other than abstracts):

Number of Non Peer-Reviewed Conference Proceeding publications (other than abstracts): 4

Peer-Reviewed Conference Proceeding publications (other than abstracts):

Number of Peer-Reviewed Conference Proceeding publications (other than abstracts): 6

(d) Manuscripts

Number of Manuscripts: 5.00

Number of Inventions:

Graduate Students

NAME	PERCENT_SUPPORTED
Xingzhou Yang	0.05
Guo Chen	0.05
Yan Gong	0.05
FTE Equivalent:	**0.15**
Total Number:	**3**

Names of Post Doctorates

NAME	PERCENT_SUPPORTED
FTE Equivalent:	
Total Number:	

Names of Faculty Supported

NAME	PERCENT_SUPPORTED	National Academy Member
Zhilin Li	0.45	No
Kazufumi Ito	0.40	No
FTE Equivalent:	**0.85**	
Total Number:	**2**	

Names of Under Graduate students supported

NAME	PERCENT_SUPPORTED
FTE Equivalent:	
Total Number:	

Student Metrics

This section only applies to graduating undergraduates supported by this agreement in this reporting period

The number of undergraduates funded by this agreement who graduated during this period: ......

The number of undergraduates funded by this agreement who graduated during this period with a degree in science, mathematics, engineering, or technology fields:......

The number of undergraduates funded by your agreement who graduated during this period and will continue to pursue a graduate or Ph.D. degree in science, mathematics, engineering, or technology fields:......

Number of graduating undergraduates who achieved a 3.5 GPA to 4.0 (4.0 max scale): ......

Number of graduating undergraduates funded by a DoD funded Center of Excellence grant for Education, Research and Engineering: ......

The number of undergraduates funded by your agreement who graduated during this period and intend to work for the Department of Defense ......

The number of undergraduates funded by your agreement who graduated during this period and will receive scholarships or fellowships for further studies in science, mathematics, engineering or technology fields: ......

Names of Personnel receiving masters degrees

NAME
Dhruba Chandru
Yi Chen
Hyoungmin Bae

Total Number: 3

Names of personnel receiving PHDs

NAME
Xingzhou Yang
Guo Chen
Yaw Kaw
Xiaohai Wan

Total Number: 4

Names of other research staff

NAME	PERCENT_SUPPORTED

FTE Equivalent:

Total Number:

Sub Contractors (DD882)

Inventions (DD882)

1 Statement of the problem studied

This proposal concerns modeling, analysis, and computational simulations of some non-linear interface problems and applications that, we believe, are of interests to DOD/ARO missions especially in the fields of automatic target recognition (ATR), computational geometry, numerical methods for stochastic differential equations, computational fluid mechanics, non-linear dynamics, imaging processing, and inverse scattering.

Most of our previous work has been focused on linear interface problems with discontinuities. We have been focusing on non-linear interface problems and applications that are of interests of DOD/ARO during the course of the project.

2 Summary of the most important results

We have successfully developed the immersed interface method for several non-linear interface problems including Poisson-Boltzmann equations on irregular domains, some variational problems, augmented approach for interface problems and problems on irregular domains.

2.1 Immersed Interface Method for a non-linear modeling of MR-fluid

The MR-fluid model is

$$\nabla \cdot (\beta(\mathbf{x}, |\nabla u|)\nabla u) = f. \tag{1}$$

The $\beta(\mathbf{x}, |\nabla u|)$ represents a nonlinear constitutive law and depends on the local material property and composition. For example, in the application of magneto-rhological (MR) fluid, say there are iron particles within a fluid, using a nonlinear saturation model, u is the magnetic potential and the empirical constitutive law (Frohlich-Kennelly relation) of the magnetic permeability of the inclusion is given by (in non-dimensionized form)

$$\beta(\mathbf{x}, H) = \begin{cases} 1 + \dfrac{(\mu - 1)\, B^{sat}}{(\mu - 1)\, H + B^{sat}} & \text{if } \mathbf{x} \text{ is within the iron,} \\ \beta^{+} & \text{if } \mathbf{x} \text{ is within fluid,} \end{cases} \tag{2}$$

where B^{sat} is a maximum attainable magnetization of the inclusion, β^{+} is a constant, and μ is the permeability of iron.

Across the interface, we have the natural jump conditions if there is no source or sink present, that is

$$[u] = 0, \qquad [\beta u_n] = 0. \tag{3}$$

where $u_n = \frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}$ is the normal derivative of the solution u, $\mathbf{n}$ is the unit normal derivative.

For different problems, the non-linear constitutive law has different forms. It is impractical to discuss them all in a single paper. Therefore, in this paper, we will explain our method particularly for the problem (1)-(3).

Finite element methods based on body-fitted grid and the boundary integral integral method has been developed for the linear model. Because the coefficient of (1) depends on the solution, it is difficult to use a boundary integral or integral equation method even for the linearized problem.

We have developed a finite difference method to solve the non-linear problem based on a modified substitution method in [5]. The linearized (1)-(3) problem is solved using the black solver of the maximum principle preserving scheme. All the discretizations are done with second order method. Global second order accuracy in the maximum norm is obtained for a problem of which we know the exact solution although it is almost impossible to prove it theoretically.

2.2 Theoretical and numerical analysis of the weighted minimal surface problem

The weighted minimal surface problem in piecewise smooth media. The solution to the weighted minimal surface problem is continuous but the derivatives have a jump across the interface where the medium property is discontinuous. The jump condition of the derivatives has been derived generalized the Snell's law in geometric optics to weighted minimal surfaces of co-dimension one in any number of dimensions. We have developed a numerical method based on the gradient flow and the maximum principal preserving immersed interface method to solve this nonlinear elliptic problem with jump conditions in [12].

2.3 A finite difference method and analysis for 2D nonlinear Poisson-Boltzmann equations

A fast finite difference method based on the monotone iterative method and the fast Poisson solver for a 2D nonlinear Poisson-Boltzmann equation is proposed and analyzed. Each iteration of the monotone method involves the solution of a linear equation in an exterior domain with an arbitrary interior boundary. A fast immersed interface method for Helmholtz equations on exterior irregular domains is used to solve the linear equation. The monotone iterative method leads to a sequence which converges monotonically from either above or below to a unique solution of the problem. This monotone convergence guarantees the existence and uniqueness of a solution as well as the convergence of the finite difference solution to the continuous solution. We also compare our results with available data in the literature to validate the numerical method. Our method is efficient in terms of accuracy, speed, and flexibility in dealing with the geometry of the domain.

2.4 Augmented methods for solving PDEs with discontinuities and problems on irregular domains

There are at least two reasons to use augmented strategies [7]. The first one is to get faster algorithms, particularly, to take advantages of existing fast solvers. The second reason is that, for some interface problems, an augmented approach may be the only way to derive an accurate algorithm. Using an augmented approach, one or several quantities of co-dimension one are introduced. The GMRES iterative method is often used to solve the augmented variable(s) that are only defined along the interface or the irregular boundary. Our recent work supported by the ARO grant including the augmented method for pressure boundary condition of the Stokes flow with a non-slip boundary condition [10], with discontinuous viscosity [9, 15], the bi-harmonic equations [1].

2.5 Other contributions

Please see the list of publications of and other reported information in the on-line form of the final report.

Our research results are summarized in the recent published book [8] in the SIAM Frontier Book Series. We have developed high order immersed interface method (IIM) in [4]. The immersed interface method has been successfully applied to variety linear and non-linear problems. It has been applied to a elasticity systems in [14, 17, 3]; Stokes and Navier Stokes equations with interfaces and singular sources in [13, 16]. Other theoretical and numerical analysis can be found in [11, 6, 2, 10].

In a summary, we are able to accomplish almost all the projective in our original proposal and more. Furthermore, the ARO grant really helped us to get our research projects going and move to new research areas. We really appreciate ARO for the support.

References

[1] G. Chen, Z. Li, and P. Lin. A fast finite difference method for biharmonic equations on irregular domains. CRSC-TR04-09, North Carolina State University, 2004.

[2] S. Deng, K. Ito, and Z. Li. Three dimensional elliptic solvers for interface problems and applications. *J. Comput. Phys.*, 184:215–243, 2003.

[3] Y. Gong and Z. Li. A conforming piecewise quadratic immersed finite element method for elliptic interface problems. Preprint, 2005.

[4] K. Ito, Y. Kyei, and Z. Li. Higher-order, Cartesian grid based finite difference schemes for elliptic equations on irregular domains. *SIAM J. Sci. Comput.*, 27:346–367, 2005.

[5] K. Ito and Z. Li. Solving a nonlinear problem in magneto-rheological fluids using the immersed interface method. *J. of Scientific Computing*, 19:253–266, in press, 2003.

[6] M-C. Lai, Z. Li, and X-B. Lin. Fast solvers for 3D Poisson equations involving interfaces in a finite or the infinite domain. *J. Comput.Appl. Math.*, 191:106–125, 2006.

[7] Z. Li. An overview of the immersed interface method and its applications. *Taiwanese J. Mathematics*, 7:1–49, 2003.

[8] Z. Li and K. Ito. *The Immersed Interface Method – Numerical Solutions of PDEs Involving Interfaces and Irregular Domains.* SIAM Frontier Series in Applied mathematics, FR33, 2006.

[9] Z. Li, K. Ito, and M-C. Lai. An augmented approach for Stokes equations with a discontinuous viscosity and singular forces, 2006.

[10] Z. Li, K. Ito, and X. Wan. Pressure jump conditions for Stokes equations with discontinuous viscosity in 2d and 3d. *Methods and Applications of Analysis*, page in press, 2006.

[11] Z. Li, T. Lin, and X. Wu. New Cartesian grid methods for interface problem using finite element formulation. *Numer. Math.*, 96:61–98, 2003.

[12] Z. Li, X. Lin, M. Torres, and H. Zhao. Generalized Snell's law for weighted minimal surface. *Methods and Applications of Analysis*, 10:199–214, 2003.

[13] Z. Li and C. Wang. A fast finite difference method for solving Navier-Stokes equations on irregular domains. *J. of Commu. in Math. Sci.*, 1:180–196, 2003.

[14] Z. Li and X. Yang. An immersed finite element method for elasticity equations with interfaces. In *Proc. Symp. Appl. Math.* Z-C. Shi et. al., editors. AMS, 2004.

[15] X. Wan, Z. Li, and S. Lubkin. An augmented approach for two-phase Stokes equations using a level set method. preprint, 2005.

[16] J. Xu, Z. Li, J. S. Lowengrub, and H. Zhao. A level-set method for interfacial flows with surfactant. *J. Comput. Phys.*, 212:590–616, 2006.

[17] X. Yang, B. Li, and Z. Li. The immersed interface method for elasticity problems with interface. *Dynamics of Continuous, Discrete and Impulsive Systems.*, 10:783–808, 2003.